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On numerical positivity of ample vector bundles with additional condition

By

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§ 0. Introduction.

In this paper these are two main theorems (Theorem 2. 1 and Theorem 3. 1). As a corollasy of these theorems, we have the following result.

Corollary 3. 7. *Let X be a non-singular projective variety of dimension m defined over the complex number field $\mathbf{C}$, and let E be a vector bundle on X of rank r. Suppose that E is ample and that, in addition, E is generated by its global sections. Then E is numerically positive.*

As for the definitions of ample vector bundles and of numerically positive vector bundles, see Definition 1. 10 and Definition 1. 9 in § 1.

In § 1, we recall definitions and some properties of such notions as invariant polynomials, positive polynomials, Chern cohomology classes, numerically positive vector bundles, and ample vector bundles. And we list notations concerning to Grassmann varieties, to Schubert cycles, and to flag manifolds. In § 2, we prove the following theorem.

Theorem 2. 1. *Let $\Theta_{\check{S}}$ be the curvature matrix of the dual vector bundle $\check{S}$ of the universal subbundle S on $Gr(n, d)$. $P(T)$ be a homogeneous polinomial in $\Pi(d+1)$ of degree q. Then the cohomology class of $P\left(\frac{\sqrt{-1}}{2\pi}\Theta_{\check{S}}\right)$ can be expressd in the form:*

the cohomology class of $P\left(\frac{\sqrt{-1}}{2\pi}\Theta_{\check{S}}\right)$

$=$ the cohomology class of $\sum_{a_0+a_1+\cdots+a_d=q}\alpha_{a_0,a_1,\cdots,a_d}\omega_{a_0,a_1,\cdots,a_d}$, where every coefficient $\alpha_{a_0,a_1,\cdots,a_d}\geqslant 0$.

We use some Schubert calculus in proving Theorem 2. 1. In § 3, we prove another theorem.

Theorem 3. 1. *Let X be a subvariety of $Gr(n, d)$ of dimension m. Assume that $X\cdot\omega_{a_0,a_1,\cdots,a_d}=0$ for some Schubert cycle $\omega_{a_0,a_1,\cdots,a_d}$ of codimension $\sum_{i=0}^{d}a_i\leqslant m$. Then there exists a curve C contained in X such that $S|C$ has* *a trivial line bundle as a direct summand, where S is the universal subbundle on $Gr(n, d)$.*

Theorem 3.1 is proved by using dimension-theoretic method. Combining Theorem 2.1 and Theorem 3.1, we prove Corollary 3.7.

In the case of line bundles, there is a famous and useful numerical criterion for ampleness proved by Y. Nakai. In the cass of vector bundles on curves, by using unitary representation of stable vector bundles, R. Hartshorne showed that ampleness is equivalent to numerical positiveness ([6]). In these two cases, the notion of the cone of positive polynomials in Chern classes is obvious. In the general case, P. A. Griffiths made a definition of the cone of positive polynomials in his article [4] (see also § 1). By using resolution of singularities and strong Lefschetz theorem, S. Bloch and D. Gieseker showed in their article [1] and in the article [3] by Gieseker that monimials in Chern classes of E and such polynomials as $\tilde{c}_q(E)$ are positive for an ample vector bundle E (for $\tilde{c}_q(E)$ see Remark 1.8 in § 1). Note that, in the case of vector bundles on 2-dimensional varieties, the above result of Bloch and Gieseker covered the whole positive polynomials. On the other hand, W. Fulton constructed recently an example of a vector bundle on $\boldsymbol{P}^2$ of rank 2, which is numerically positive but not ample ([2]). Hence, the remaining problem is to see whether our additional assumption in Corollary 3.7 can be removed.

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§ 1. Definitions and notations.

a. Invariant polynomials and positive polynomials.

In this subsection, we follow Griffiths [4]. Let $M_{\boldsymbol{Q}}(d+1)$ be the affine space consisting of all $((d+1)\times(d+1))$-matrices over the rational number field $\boldsymbol{Q}$, and let $A(d+1)$ be the coordinate ring of $M_{\boldsymbol{Q}}(d+1)$. We denote by

$$T=\begin{bmatrix} T_{00}T_{01}\cdots T_{0d} \\ \cdots\cdots\cdots \\ T_{d0}T_{d1}\cdots T_{dd}\end{bmatrix}$$

an indeterminat $(d\times 1)\times(d+1)$-matrix, i.e. T_{ij} $(0\leqslant i, j\leqslant d)$ form an affine coordinates of $M_{\boldsymbol{Q}}(d+1)$. Then, $A(d+1)=\boldsymbol{Q}[T_{00}, T_{01}, \cdots, T_{dd}]$. $GL_{\boldsymbol{Q}}(d+1)$ acts naturally on $M_{\boldsymbol{Q}}(d+1)$, i.e. for $g\in GL_{\boldsymbol{Q}}(d+1)$ and $a\in M_{\boldsymbol{Q}}(d+1)$, $a\mapsto g^{-1}ag$, where the product is the product as matrices.

Definition 1.1. *A polynomial $P(T)$ in $A(d+1)$ is said to be an invariant polynomial if $P(T)$ is $GL_{\boldsymbol{Q}}(d+1)$-invariant under the above action. We denote by $I(d+1)$ the subalgebra of $A(d+1)$, consisting of all invariant polynomials, and call it an invariant polynomial ring.*

Let Δ_q be the principal q-minar determinant of T $(1\leqslant q\leqslant d+1)$, i.e.

$$\Delta_q = \sum_{0 \leqslant i_1 < i_2 < \cdots < i_q \leqslant d} \det \begin{bmatrix} T_{i_1 i_1} T_{i_1 i_2} \cdots T_{i_1 i_q} \\ \cdots\cdots\cdots\cdots \\ T_{i_q i_1} T_{i_q i_2} \cdots T_{i_q i_q} \end{bmatrix}$$

Let $\boldsymbol{Q}[T_0, T_1, \cdots, T_d]$ be a polynomial ring in $d+1$ indeterminates over the field of rational numbers. Let $\mathfrak{S}_{d+1}$ be the symmetric group over $d+1$ elements. $\mathfrak{S}_{d+1}$ acts naturally on $\boldsymbol{Q}[T_0, T_1, \cdots, T_d]$ by permuting the order of indeterminates. We denote by $\boldsymbol{Q}[T_0, T_1, \cdots, T_d]^{\mathfrak{S}_{d+1}}$ the subalgebra of $\boldsymbol{Q}[T_0, T_1, \cdots, T_d]$ consisting of $\mathfrak{S}_{d+1}$-invariant elements, i.e. symmetric polynomials in $T_0, T_1, \cdots, T_d$. Let S_q be the fundamental symmetric form in $T_0, T_1, \cdots, T_d$ of degree q $(1 \leqslant q \leqslant d+1)$.

Lemm 1. 2. *We have the following interpretation of an invariant polynomial ring* $I(d+1)$.

$$\begin{array}{ccc} I(d+1) & \xleftarrow[\sim]{f} & \boldsymbol{Q}[T_0, T_1, \cdots, T_d]^{\mathfrak{S}_{d+1}} \\ g \wr \uparrow & & \wr \uparrow g' \\ \boldsymbol{Q}[\Delta_1, \Delta_2, \cdots, \Delta_{d+1}] & \xleftarrow[f']{\sim} & \boldsymbol{Q}[S_1, S_2, \cdots, S_{d+1}], \end{array}$$

where g and g' are natural inclusion maps, f is defined by $f(T_i) = T_{ii}$ $(0 \leqslant i \leqslant d)$, and f' is defined by $f(S_q) = \Delta_q$ $(1 \leqslant q \leqslant d+1)$. Moreover, if we weight Δ_q's and S_q's with $\deg \Delta_q = q$ and with $\deg S_q = q$ $(1 \leqslant q \leqslant d+1)$, all the maps are isomorphisms as graded algebras.

Proof. It is obvious that f' and g' are isomorphisms and that g is an injection. Hence, it is enough to show that f is surjective. Define a map $h: I(d+1) \to \boldsymbol{Q}[T_0, T_1, \cdots, T_d]^{\mathfrak{S}_{d+1}}$ by $h(T_{ij}) = \delta_{ij} T_i$ $(0 \leqslant i, j \leqslant d)$. It is easy to see that h is well-defined. Note that $\operatorname{Spec}(I(d+1)) = M_{\boldsymbol{Q}}(d+1)/GL_{\boldsymbol{Q}}(d+1)$. Let $(f \circ h)^*$ be the corresponding endomorphism of $\operatorname{Spec}(I(d+1))$. Since $(f \circ h)^*$ is an automorphism on the classes of diagonaizable matirces and since the classes of diagonaizable matrices is an open dense subset of $M_{\boldsymbol{Q}}(d+1)/GL_{\boldsymbol{Q}}(d+1)$, we see that $(f \circ h)^*$ is an automorphism on $\operatorname{Spec}(I(d+1))$. Hence, $f \circ h$ is an automorphism of $I(d+1)$. Therefore, f is surjective. Q.E.D.

Corollary 1. 3. *An invariant homogeneous polynomial $P(T)$ of degree q in $I(d+1) \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ can be expressed (not necessarilly unique) in the form*

$$P(T) = \sum_{\begin{cases} \rho \in [0, d]^q \\ \pi, \tau \in \mathfrak{S}_q \end{cases}} p_{\rho\pi\tau} T_{\rho_{\pi 1} \rho_{\tau 1}} \cdots T_{\rho_{\pi q} \rho_{\tau q}},$$

where $p_{\rho\pi\tau} \in \boldsymbol{C}$ $(\rho \in [0, d]^q$, π and $\tau \in \mathfrak{S}_q)$.

Proof. Our assertion follows from the following three facts:

(1) $P(T) \in \boldsymbol{C}[\Delta_0, \Delta_1, \cdots, \Delta_d]$.

(2) $\Delta_r(T) = \left(\frac{1}{r!}\right) \sum_{\begin{cases} \rho \in [0, d]^r \\ \pi, \tau \in \mathfrak{S}_r \end{cases}} \operatorname{sgn} \pi \operatorname{sgn} \tau T_{\rho_{\pi 1} \rho_{\tau 1}} \cdots T_{\rho_{\pi r} \rho_{\tau r}}$.

(3) *Let* $\mathscr{T}=\left\{P(T)\in A(d+1)\underset{Q}{\otimes}C \;\middle|\; \begin{array}{l}P(T) \text{ has an expression of the type}\\ \text{stated in Corollary 1.3}\end{array}\right\}$

be a subset of $A(d+1)\otimes_{Q}C$. Then, $\mathscr{T}$ forms a graded subring of $A(d+1)\otimes_{Q}C$.

As for (3), by definition, $\mathscr{T}$ is closed under addition and under subtracion. We have to see that $\mathscr{T}$ is also closed under multiplication. Let

$$P(T)=\sum_{\substack{\rho\in[0,d]^q\\ \pi,\tau\in\mathfrak{S}_q}} p_{\rho\pi\tau}T_{\rho_{\pi1}\rho_{\tau1}}\cdots T_{\rho_{\pi q}\rho_{\tau q}} \quad \text{and}$$

$$P'(T)=\sum_{\substack{\rho'\in[0,d]^{q'}\\ \pi',\tau'\in\mathfrak{S}_{q'}}} p'_{\rho'\pi'\tau'}T_{\rho'_{\pi'1}\rho'_{\tau'1}}\cdots T_{\rho'_{\pi'q'}\rho'_{\tau'q'}}$$

be homogeneous elements in $\mathscr{T}$ of degree q and of degree q' respectively. Then, we have

$$P(T)P'(T)=\sum_{\substack{\rho\in[0,d]^q\\ \rho'\in[0,d]^{q'}\\ \pi,\tau\in\mathfrak{S}_q\\ \pi',\tau'\in\mathfrak{S}_{q'}}} p_{\rho\pi\tau}p'_{\rho'\pi'\tau'}T_{\rho_{\pi1}\rho_{\tau1}}\cdots T_{\rho_{\pi q}\rho_{\tau q}}T_{\rho'_{\pi'1}\rho'_{\tau'1}}\cdots T_{\rho'_{\pi'q'}\rho'_{\tau'q'}}$$

Put $\rho''=(\rho,\rho')\in[0,d]^{q+q'}$, $\tilde{\pi}=(\pi,\pi')$ and $\tilde{\tau}=(\tau,\tau')\in\mathfrak{S}_q\times\mathfrak{S}_{q'}$, $\tilde{p}_{\rho''\tilde{\pi}\tilde{\tau}}=p_{\rho\pi\tau}p_{\rho'\pi'\tau'}$. Fix a left coset decomposition of $\mathfrak{S}_{q+q'}$ by $\mathfrak{S}_q\times\mathfrak{S}_{q'}$, say, $\mathfrak{S}_{q+q'}=\coprod_\alpha\omega_\alpha\mathfrak{S}_q\times\mathfrak{S}_{q'}$. Then, for each π'' in $\mathfrak{S}_{q+q'}$, there exists a unique ω_α such that $\omega_\alpha^{-1}\pi''\in\mathfrak{S}_q\times\mathfrak{S}_{q'}$. Hence, for example, taking $p''_{\rho''\pi''\tau''}=\left(\dfrac{q!q'!}{(q+q')!}\right)^2\tilde{p}_{\rho,\omega_\alpha^{-1}\pi'',\omega_\beta^{-1}\tau''}$, we have

$$P(T)P'(T)=\sum_{\substack{\rho''\in[0,d]^{q+q}\\ \pi'',\tau''\in\mathfrak{S}_{q+q}}} p''_{\rho''\pi''\tau''}T_{\rho''_{\pi''1}\rho''_{\tau''1}}\cdots T_{\rho''_{\pi''(q+q')}\rho''_{\tau''(q+q')}}.$$

Q.E.D.

Definition 1.4, (*Griffiths*). *An invariant homogeneous polynomial $P(T)$ of degree q in $I(d+1)\otimes_{Q}C$ is said to be a positive polynomial if there exist $\lambda_{\rho j}\in\boldsymbol{R}$ with $\lambda_{\rho j}>0$ and $\mu_{\rho j\pi}\in\boldsymbol{C}$ ($\rho\in[0,d]_q$, $\pi\in\mathfrak{S}_q$, and j runs over a finite set) such that*

$$p_{\rho\pi\tau}=\sum_j\lambda_{\rho j}\mu_{\rho j\pi}\bar{\mu}_{\rho j\tau},$$

for each coefficient $p_{\pi\tau\rho}$ in the expression of $P(T)$ mentioned in Corollary 1.3. We denote by $\Pi(d+1)_q$ the set of all homogeneous positive polynomials of degree q, and put $\Pi(d+1)=\sum_q\Pi(d+1)_q$.

Lemma 1.5. *The set $\Pi(d+1)$ of positive polynomials is closed under addition and under multiplication.*

Proof. By definition, $\Pi(d+1)$ is closed under addition. We can see that $\Pi(d+1)$ is closed under multipication in the same way as the proof of Corollary 1.3. Q.E.D.

b. Chern cohomology classes.

Let X be a non-singular complete variety defined over the complex number field $\boldsymbol{C}$. Let E be a vector bundle of rank $d+1$ on X, and let Θ_E be a curvature matrix of E.

Lemma 1.6, (*Weil homomorphism*). *The map* $w_E: I(d+1)\otimes_{\boldsymbol{Q}}\boldsymbol{C}\to H^{\cdot}(X,\boldsymbol{C})$ *defined by* $P(T)\mapsto\left(\text{the cohomology class of } P\left(\frac{\sqrt{-1}}{2\pi}\Theta_E\right)\right)$ *is well-defined and is a homomorphism as graded algebras. Moreover, the map* w_E *is functorial, i.e. for a morphism* $g: Y\to X$, *we have* $g^*\circ w_E=w_{g^*E}$.

Proof. See Weil [10]. Q.E.D.

Definition 1.7. *The cohomology class of* $\Delta_q\left(\frac{\sqrt{-1}}{2\pi}\Theta_E\right)$ *is called the q-th Chern cohomology class of E.*

Let R be a commutative ring with identity and let $R[[t]]$ be the formal power series ring of one variable t with coefficients in R. For each element $c(t)=1+c_1t+c_2t^2+\cdots$ in $R[[t]]$, we define an element $\tilde{c}(t)=1+\tilde{c}_1t+\tilde{c}_2t^2+\cdots$ in $R[[t]]$ by $\tilde{c}(t)=1/c(-t)$. We can calculate $\tilde{c}_q$'s successively such as

$$\tilde{c}_1=c_1,$$
$$\tilde{c}_2=c_1^2-c_2,$$
$$\tilde{c}_3=c_1^3-2c_1c_2+c_3, \text{ etc.}$$

Remark 1.8. *If we take* $R=I(d+1)$ *and* $c_q=\begin{cases}\Delta_q(T) & 1\leqslant q\leqslant d+1\\ 0 & d+1<q,\end{cases}$ *then we have*

$$\tilde{c}_q=\sum_{\alpha_0+\cdots+\alpha_d=q}T_0^{\alpha_0}\cdots T_d^{\alpha_d}\in\boldsymbol{Q}[T_0,\cdots,T_d]^{\mathfrak{S}_{d+1}}=I(d+1)\quad(q=1,2,\cdots).$$

Note that such $\tilde{c}_q=\sum_{\alpha_0+\cdots+\alpha_d=q}T_0^{\alpha_0}\cdots T_d^{\alpha_d}$ *is a positive polynomial* $(q=1,2,\cdots)$.

c. Numerically positive vector bundles and ample vector bundles.

Let X be a non-singular projective variety of dimension m defined over $\boldsymbol{C}$. Let E be a vector bundle on X.

Definition 1.9, (*Griffiths*). *A vector bundle on X is said to be numerically positive if it satisfies the following condition.*

Let q be any integer with $1\leqslant q\leqslant m=\dim X$. *Let Y be any q-dimensional subvariety of X. Let F be any quotient vector bundle of rank* $d+1$ *of E with* $F\neq 0$. *Let* $P(T)$ *be any homogeneous positive polynomial in* $\Pi(d+1)$ *of degree q with* $P(T)\neq 0$. *Then we have*

$$\int_Y P\left(\frac{\sqrt{-1}}{2\pi}\Theta_F\right)>0.$$

For the properties of numerically positive vector bundles, see Griffiths [4].

Let X be a non-singular projective variety defined over a field k of any characteristic. Let E be a vector bundle on X.

Definition 1.10, (*Hartshorne*). *A vector bundle E on X is said to be ample if, for any coherent sheaf M on X, $M \otimes S^a(E)$ is generated by its global sections for a sufficiently large integer a, where $S^a(E)$ is the a-th symmetric product of E.*

For the properties of ample vector bundles, see Hartshorne [5].

d. Grassmann varieties, Schubert cycles, and flag manifolds.

We use the following notations throughout this paper.

$Gr(n, d)$: the Grassmann variety parametrizing d-dimensional linear subspaces of $\boldsymbol{P}^n$.

L_x : the d-dimensional linear subspace of $\boldsymbol{P}^n$ corresponding to a point x in $Gr(n, d)$.

S : the universal subbundle on $Gr(n, d)$.

Q : the universal quotient bundle on $Gr(n, d)$.

$Fl(n; d_1, \cdots, d_e)$: the flag manifold parametrizing filtrations of $\boldsymbol{P}^n$ by linear subspaces of $\boldsymbol{P}^n$.

Let n and d be non-negative integers with $n \geqslant d$, and let a_i $(0 \leqslant i \leqslant d)$ be integers with $n-d \geqslant a_0 \geqslant a_1 \geqslant \cdots \geqslant a_d \geqslant 0$. Take a filtration

$$A_{n-d-a_0} \subset A_{n-d+1-a_1} \subset \cdots \subset A_{n-a_d}$$

of $\boldsymbol{P}^n$ by linear subspaces $A_{n-d+i-a_i}$'s of $\boldsymbol{P}^n$ with $\dim A_{n-d+i-a_i} = n-d+i-a_i$. We use the following notations for Schubert varieties and Schubert cycles

$$\begin{aligned} &\omega_{a_0, a_1, \cdots, a_d}(A_{n-d-a_0}, A_{n-d+1-a_1}, \cdots, A_{n-a_d}) \\ &\quad = \{x \in Gr(n, d) \mid \dim(L_x \cap A_{n-d+i-a_i}) \geqslant i \quad (0 \leqslant i \leqslant d)\}. \end{aligned}$$

$\omega_{a_0, a_1, \cdots, a_d}$: the Schubert cycle on $Gr(n, d)$ of type $(a_0, a_1, \cdots, a_d)$. Note that $\operatorname{codim}_{Gr(n,d)} \omega_{a_0, a_1, \cdots, a_d} = \sum_{i=0}^d a_i$.

§ 2. Positive polynomials and Schubert cycles.

We have an exact sequence of vector bundles on $Gr(n, d)$

$$0 \to S \to F \to Q \to 0,$$

where S is the universal subbundle, F the trivial bundle, and Q the universal quotient bundle. Let h be the trivial hermitian metric in F, and let $f(z) = (e_0(z), \cdots, e_d(z), \cdots, e_n(z))$ be a local unitary frame of F with respect to h so that $\varphi(z) = (e_0(z), \cdots, e_d(z))$ is a local frame of S. We denote by D_F the connection of F derived from h and by D_S the connection of S derived from the induced metric in S. Then we have the following diagram

$$\begin{tikzcd} A^0(S) \arrow[r, "D_S"] \arrow[d, hook, "i^0"'] & A^1(S) \arrow[d, hook, "i^1"'] \\ A^0(F) \arrow[r, "D_F"] & A^1(F). \end{tikzcd}$$

Griffiths calls the gap $D_F \circ i^0 - i^1 \circ D_S$ the second fundamental form of S in F

([4]). We use the following notations.

$\theta_F=\begin{bmatrix}\theta_1 & \theta_2\\ \theta_3 & \theta_4\end{bmatrix}$: the connection matrix of D_F with respect to the frame $f(z)$.

θ_S : the connection matrix of D_S with respect to the frame $\varphi(z)$.

b : the matrix representation of $D_F\circ i^0-i^1\circ D_S$ with respect to the frames $\varphi(z)$ and $f(z)$.

$\Theta_F=\begin{bmatrix}\Theta_1 & \Theta_2\\ \Theta_3 & \Theta_4\end{bmatrix}$: the curvature matrix of D_S with respect to the frame $f(z)$.

Θ_S : the curvature matrix of D_S with respect to the frame $\varphi(z)$.

$\Theta_{\check{S}}=-{}^t\Theta_S$: the induced curvature matrix of the dual bundle $\check{S}$ of S.

Now we prove the following theorem.

Theorem 2.1. *Let $\Theta_{\check{S}}$ be the curvature matrix of the dual vector bundle $\check{S}$ of the universal subbundle S on $Gr(n, d)$. Let $P(T)$ be a homogeneous positive polynomial in $\Pi(d+1)$ of degree q. Then the cohomology class of $P\left(\frac{\sqrt{-1}}{2\pi}\Theta_{\check{S}}\right)$ can be expressed in the form:*

the cohomology class of $P\left(\frac{\sqrt{-1}}{2\pi}\Theta_{\check{S}}\right)$

$=$the cohomology class of $\sum_{a_0+a_1+\cdots+a_d=q}\alpha_{a_0,a_1,\cdots,a_d}\omega_{a_0,a_1,\cdots,a_d}$, where every coefficient $\alpha_{a_0,a_1,\cdots,a_d}\geqslant 0$.

i.e. In the $\boldsymbol{R}$-vector space $H^{2q}(Gr(n,d),\boldsymbol{R})$ we have the following inclusion of cones.

$$\left\{\begin{array}{l|l}\text{the cohomo-} & P(T)\text{ is a homogeneous}\\ \text{logy class of} & \text{positive polynomial in}\\ P\left(\frac{\sqrt{-1}}{2\pi}\Theta_{\check{S}}\right) & \Pi(d+1)\text{ of degree } q\end{array}\right\}\subset\begin{bmatrix}\text{the cone generated by}\\ \text{Schubert cycles on}\\ Gr(n,d)\text{ of codimension } q\end{bmatrix}.$$

Lemma 2.2. *$\theta_S=\theta_1$, $b=\theta_3$, and b is a matrix consisting of $(1,0)$-forms.*

Proof. By definitions, we have $\theta_S=\theta_1$ and hence $b=\theta_3$. In order to prove the third assertion, take a holomorphic frame $\tilde{\varphi}(z)=(\tilde{e}_0(z),\cdots,\tilde{e}(z))$ of S. Let $\tilde{\varphi}(z)=\varphi(z)g(z)$ be the change of frames. Let $D_F=D_F'+D_F''$ be the type decomposition. Since D_F'' is 0 on the holomorphic sections of F by definition, we have

$$\begin{aligned}(0,\cdots,0)&=(D_F''e_0(z),\cdots,D_F''e_d(z))\\ &=(e_0(z),\cdots,e_n(z))\left(\begin{bmatrix}d''g(z)\\ 0\end{bmatrix}+\begin{bmatrix}\theta_1'' & \theta_2''\\ \theta_3'' & \theta_4''\end{bmatrix}\begin{bmatrix}g(z)\\ 0\end{bmatrix}\right).\end{aligned}$$

Hence, we have $\theta_3''=0$, i.e. θ_3 is of type $(1,0)$ Q.E.D.

Corollary 2.3. $\Theta_S={}^t\bar{b}\wedge b$.

Proof. $\Theta_F = d\theta_F + \theta_F \wedge \theta_F$

$$= \begin{bmatrix} d\theta_1 + \theta_1 \wedge \theta_1 + \theta_2 \wedge \theta_3 & d\theta_2 + \theta_1 \wedge \theta_2 + \theta_2 \wedge \theta_4 \\ d\theta_3 + \theta_3 \wedge \theta_1 + \theta_4 \wedge \theta_3 & d\theta_4 + \theta_3 \wedge \theta_2 + \theta_4 \wedge \theta_4 \end{bmatrix}.$$

By the compatibility of D_F and h we have

$$0 = {}^t\bar{\theta}_3 + \theta_2.$$

Therefore, by using Lemma 2. 2, we have

$$\Theta_1 = \Theta_S - {}^t\bar{b} \wedge b.$$

Since F is a trivial bundle, $\Theta_F = 0$. Hence, we have

$$\Theta_S = {}^t\bar{b} \wedge b. \qquad \text{Q.E.D.}$$

Lemma 2. 4. *Let $g = \dim Gr(n,d)$. $H^{2q}(Gr(n,d), \boldsymbol{R})$ and $H^{2g-2q}(Gr(n,d), \boldsymbol{R})$ are dual for cup product pairing and the Schubert cycles*

$$\{\omega_{a_0, a_1, \cdots, a_d} \mid a_0 + a_1 + \cdots + a_d = q\} \text{ and}$$
$$\{\omega_{b_0, b_1, \cdots, b_d} \mid b_0 + b_1 + \cdots + b_d = g - q\}$$

form the dual base each other.

i.e. $\omega_{a_0, a_1, \cdots, a_d} \cdot \omega_{b_0, b_1, \cdots, b_d} = \begin{cases} 1 & a_i + b_j = n - d \ (i + j = d), \\ 0 & \text{otherwise.} \end{cases}$

Proof. See Hodge & Pedoe [7]. Q.E.D.

Proof of Theorem 2. 1. By Corollary 2. 3, we have

$$\Theta_S^{\check{}} = -{}^t\Theta_S = -{}^t({}^t\bar{b} \wedge b) = {}^t\bar{b} \wedge b.$$

Let $P(T) = \sum_{\substack{\rho \in [0, d]^q \\ \pi, \tau \in \mathfrak{S}_q}} \mu_{\rho\pi} \bar{\mu}_{\rho\tau} T_{\rho_{\pi 1} \rho_{\tau 1}} \cdots T_{\rho_{\pi q} \rho_{\tau q}}$ be a positive polynomial of degree q. Substituting $\Theta_S^{\check{}}$ in $P(T)$, we have

$$\begin{aligned} P(\Theta_S^{\check{}}) &= \sum_{\substack{\rho \in [0, d]^q \\ \alpha \in [d+1, n]^q \\ \pi, \tau \in \mathfrak{S}_q}} \mu_{\rho\pi} \bar{\mu}_{\rho\tau} b_{\alpha_1 \rho_{\pi 1}} \bar{b}_{\alpha_1 \rho_{\tau 1}} \cdots b_{\alpha_q \rho_{\pi q}} \bar{b}_{\alpha_q \rho_{\tau q}} \\ &= (-1)^{q(q-1)/2} \sum \mu_{\rho\pi} \bar{\mu}_{\rho\tau} b_{\alpha_1 \rho_{\pi 1}} \cdots b_{\alpha_q \rho_{\pi q}} \bar{b}_{\alpha_1 \rho_{\tau 1}} \cdots \bar{b}_{\alpha_q \rho_{\tau q}} \\ &= (-1)^{q(q-1)/2} \sum_{\rho, \alpha} Q_{\rho\alpha} \bar{Q}_{\rho\alpha}, \end{aligned}$$

where $b = \begin{bmatrix} b_{d+1\,0} \cdots b_{d+1\,d} \\ \cdots\cdots\cdots \\ b_{n\,0} \ \cdots b_{n\,d} \end{bmatrix}$ and $Q_{\rho\alpha} = \sum_{\pi \in \mathfrak{S}_q} \mu_{\rho\pi} b_{\alpha_1 \rho_{\pi 1}} \cdots b_{\alpha_q \rho_{\pi q}}$. Hence, we have

$$P\left(\frac{\sqrt{-1}}{2\pi} \Theta_S^{\check{}}\right) = (-1)^{q(q-1)/2} \left(\frac{\sqrt{-1}}{2\pi}\right)^q \sum_{\rho, \alpha} Q_{\rho\alpha} \bar{Q}_{\rho\alpha}.$$

Since, by Lemma 2. 2, b is a matrix consiting of (1, 0)-forms, Q_α's are $(q, 0)$-forms. Therefore, for any q-dimensional subvariety Y of $Gr(n, d)$, we have

$$\int_Y P\left(\frac{\sqrt{-1}}{2\pi}\Theta_S^{\vee}\right)\geqslant 0.$$

That is, the cohomology class of $P\left(\frac{\sqrt{-1}}{2\pi}\Theta_S^{\vee}\right)$ is numerically non-negative. Hence, in the expression

the cohomology class of $P\left(\frac{\sqrt{-1}}{2\pi}\Theta_S^{\vee}\right)$

$=$*the cohomology class of* $\sum_{a_0+a_1+\cdots+a_d=q}\alpha_{a_0,a_1,\cdots,a_d}\omega_{a_0,a_1,\cdots,a_d}$,

we see, by using Lemma 2. 4, that every coefficient $\alpha_{a_0,a_1,\cdots,a_d}\geqslant 0$. Q.E.D.

§ 3. Numerical positivity of ample vector bundles.

In this section, we prove the following theorem.

Theorem 3. 1. *Let X be a subvariety of $Gr(n, d)$ of dimension m. Assume that $X\cdot\omega_{a_0,a_1,\cdots,a_d}=0$ for some Schubert cycle $\omega_{a_0,a_1,\cdots,a_d}$ of codimension $\sum_{i=0}^{d}a_i\leqslant m$. Then there exists a curve C contained in X such that $S|C$ has a trivial line bundle as a direct summand, where S is the universal subbundle.*

Let X be a subvariety of $Gr(n, d)$ of dimension m. We call a point P in $\boldsymbol{P}^n$ a center of X if $\dim\{x\in X\,|\,L_x\ni P\}\geqslant 1$. We denote by X_c the set of centers of X.

Lemma 3. 2. X_c is a closed subset of $\boldsymbol{P}^n$.

Proof. By the Plücker coordinates, $Gr(n, d)$ can be embedded in a projective space $\boldsymbol{P}^N$. Put $Y_P=\{x\in X\,|\,L_x\ni P\}$ for a point P in $\boldsymbol{P}^n$. Then, for a point P in $\boldsymbol{P}^n$, it is easy to see that the following conditions are equivalent to each other:

i) $P\in X_c$,

ii) $\dim Y_P\geqslant 1$,

iii) $Y_P\cap M\neq\phi$ for any hyperplane M in $\boldsymbol{P}^N$.

Hence, we have $X_c=\bigcap_M(\bigcup_{x\in M\cap X}L_x)$. Therefore X_c is a closed subset of $\boldsymbol{P}^n$. Q.E.D.

Let X be as above and let H be a general hyperplane in $\boldsymbol{P}^n$. Let x_1 be a generic point of X. Then there exists unique point y_1 of $Gr(n, d-1)$ such that $L_{y_1}=L_{x_1}\cap H$. We denote by X_H the subvariety of $Gr(n, d-1)$ which has y_1 as a generic point. Since $\{y\in Gr(n, d-1)\,|\,L_y\subset H\}$ is isomorphic to $Gr(n-1, d-1)$, X_H can be also regarded as a subvariety of $Gr(n-1, d-1)$.

Lemma 3. 3. *Assume that $X\cdot\omega_{a_0a_1,\cdots,a_{d-1},0}=0$ on $Gr(n, d)$. Then $X_H\cdot\omega_{a_0,a_1,\cdots,a_{d-1}}=0$ on $Gr(n-1, d-1)$ for any general hyperplane H in $\boldsymbol{P}^n$.*

Proof. Since H is general and since $X\cdot\omega_{a_0,a_1,\cdots,a_{d-1},0}=0$, there exists a sequence

$$A_0\subset A_1\subset\cdots\subset A_{d-1}\subset A_d=H$$

of linear subspaces of $\boldsymbol{P}^n$ such that $\dim A_i=n-d+i-a_i$ $(0\leqslant i\leqslant d-1)$ and that

$$\{x \in X \mid \dim(L_x \cap A_i) \geqslant i \ (0 \leqslant i \leqslant d-1)\} = \phi.$$

By the definition of X_H, for any point y in X_H, there exists a point x in X such that $L_y \subset L_x$. Hence, we have

$$\{y \in X_H \mid \dim(L_y \cap A_i) \geqslant i \ (0 \leqslant i \leqslant d-1)\} = \phi,$$

that is, $X_H \cdot \omega_{a_0, a_1, \cdots, a_{d-1}} = 0$ as a cycle on $Gr(n-1, d-1)$. Q.E.D.

Lemma 3. 4. *Let X be subvariety of $Gr(n, d)$ of dimension m. Assume that $m \geqslant d+1$ and that $X \cdot \omega_{1,1,\cdots,1} = 0$. Then, for any point x in X, we have $L_x \cap X_c \neq \phi$.*

Proof. Let x_0 be a point in X. We consider the following diagram:

$$\begin{array}{ccc} & Fl(n; n-1, d) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ Gr(n, n-1) & & Gr(n, d) \supset X \ni x_0. \end{array}$$

Set $Z = \pi_1 \circ \pi_2^{-1}(x_0) = \{h \in Gr(n, n-1) \mid L_h \supset L_{x_0}\}$ and set $W = \pi_1^{-1}(Z) \cap \pi_2^{-1}(X) = \{(h, x) \in Fl(n; n-1, d) \mid x \in X, L_h \supset L_x, \text{ and } L_h \supset L_{x_0}\}$. For any point h in Z, we have, by our assumption $Y \cdot \omega_{1,1,\cdots,1} = 0$, that

$$\dim(\pi_1^{-1}(h) \cap W) = \dim(X \cap \omega_{1,1,\cdots,1}(L_h)) \geqslant \dim X - d.$$

Hence, there exists an irreducible component W_0 of W such that

(1) $\dim W_0 \geqslant \dim Z + \dim X - d = \dim X + n - 2d - 1.$

Put $Y_0 = \pi_2(W_0)$. From (1), we have, for any point x in Y_0, that

(2) $\dim(\pi_2^{-1}(x) \cap W) \geqslant \dim(\pi_2^{-1}(x) \cap W_0) \geqslant \dim X + n - 2d - 1 - \dim Y_0.$

Since $\pi_2^{-1}(x) \cap W \cong \{h \in Gr(n, n-1) \mid L_h \supset L_x \text{ and } L_h \supset L_{x_0}\}$, we have

(3) $\dim(\pi_2^{-1}(x) \cap W) = n - 1 - \dim \left\{ \begin{array}{l} \textit{linear subspace of } \boldsymbol{P}^n \\ \textit{spanned by } L_x \textit{ and } L_{x_0} \end{array} \right\}$

$$= n - 1 - (2d - \dim(L_x \cap L_{x_0})).$$

Combining (2) and (3), we have, for any point x in Y_0, that

(4) $\dim(L_x \cap L_{x_0}) \geqslant \dim X - \dim Y_0.$

Next, we consider the following diagram:

$$\begin{array}{ccc} & Fl(n; d, 0) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ Gr(n, d) \supset X \supset Y_0 & & Gr(n, 0) = \boldsymbol{P}^n \supset L_x. \end{array}$$

From (4), we have

$$\dim(p_1^{-1}(Y_0) \cap p_2^{-1}(L_{x_0})) \geqslant \dim Y_0 + \dim X - \dim Y_0 = \dim X.$$

Hence, for any point P in $p_2 \circ p_1^{-1}(Y_0) \cap L_{x_0}$, we have

$$\dim(p_1^{-1}(Y_0) \cap p_2^{-1}(P)) \geqslant \dim X - \dim L_{x_0} = m - d \geqslant 1.$$

This shows that there exists an $(m-d)$-dimensional subvariety Y of X such that, for any point x in Y, L_x goes through a common point P in $\boldsymbol{P}^n$. That is $L_{x_0} \cap X_c \neq \phi$. Q.E.D.

Lemma 3.5. *Let X be a subvariety of $Gr(n, d)$ of dimension m. Assume that $X \cdot \omega_{a_0, a_1, \cdots, a_d}=0$ for some Schubert cycle $\omega_{a_0, a_1, \cdots, a_d}$ of codimension $\sum_{i=0}^{d} a_i \leqslant m$. Then, for any point x in X, we have $L_x \cap X_c \neq \phi$.*

Proof. We prove the above statement by induction on m and on d. When $m=1$ or $d=0$, it is obvious. We now assume that $m \geqslant 2$ and $d \geqslant 1$.

Case 1. When $a_d > 0$: If $X \cdot \omega_{1,1,\cdots,1}=0$, the assertion is proved in Lemma 3.4. Therefore we may assume that $X \cdot \omega_{1,1,\cdots,1} \neq 0$. Note that $\omega_{a_0, a_1, \cdots, a_d} = \omega_{a_0-1, a_1-1, \cdots, a_d-1} \cdot \omega_{1,1,\cdots,1}$. Let H be a general hyperplane in $\boldsymbol{P}^n$. Since dim $(X \cap \omega_{1,1,\cdots,1}(H))=m-d-1$ and since $(X \cap \omega_{1,1,\cdots,1}(H)) \cdot \omega_{a_0-1, a_1-1, \cdots, a_d-1}=0$, the assertion is proved by induction hypothesis.

Case 2. When $a_d=0$: Assume that $L_{x_0} \cap X_c = \phi$, for a point x_0 in X, and and we will derive a contradiction. We use the following diagrams and notations.

(1)
$$\begin{array}{ccc} & Fl(n; d, d-1) & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle q_1} \\ Gr(n, d) \supset X & & Gr(n, d-1) \supset Gr(n-1, d-1) \supset X_H, \end{array}$$

where p_1 and q_1 are natural projections, and put

$\tilde{X} = p_1^{-1}(X) \cap q_1^{-1}(X_H)$,

$X_0 = \{x \in X \mid L_x \subset H\} = X \cap \omega_{1,1,\cdots,1}(H)$, and

$X_{H,0} = q_1 \circ p_1^{-1}(X_0)$.

(2)
$$\begin{array}{ccc} & Fl(n-1; d-1, 0) & \\ {\scriptstyle p_2} \swarrow & & \searrow {\scriptstyle q_2} \\ Gr(n-1, d-1) \supset X_H \supset X_{H,0} & & Gr(n-1, 0) = \boldsymbol{P}^{n-1} = H, \end{array}$$

where p_2 and q_2 are natural projections, and put

$Y = q_2 \circ q_2^{-1}(X_H) = \bigcup_{y \in X_H} L_y$, and

$Y_0 = q_2 \circ p_2^{-1}(X_{H,0}) = \bigcup_{y \in X_{H,0}} L_y$.

(3)
$$\begin{array}{ccccc} & & Fl(n-1; d-1, 0) \times H & & \\ & {\scriptstyle \pi_1 = p_2 \times 1_H} \swarrow & & \searrow {\scriptstyle \pi_2 = q_2 \times 1_H} & \\ & Gr(n-1, d-1) \times H & & H \times H & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle q} \quad {\scriptstyle r} \swarrow & & \searrow {\scriptstyle s} \\ Gr(n-1, d-1) \supset X_H & & H \supset Y_0 & & H \supset Y_0, \end{array}$$

where the projections are defined by

$$\begin{tikzcd}[column sep=small] & & & (y,Q,P) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & & \\ & & (y,P) \arrow[dll, "p"'] \arrow[dr, "q"] & & (Q,P) \arrow[dl, "r"'] \arrow[drr, "s"] & & \\ y & & & P & & & Q. \end{tikzcd}$$

We go on in several steps.

Step 1. $\dim X_H = \dim \tilde{X} = \dim X$.

Indeed, working with the diagram (1), we have a point y_0 in X_H with $L_{y_0} \subset L_{x_0} \cap H$. By the assumption $L_{x_0} \cap X_c = \phi$, we see that $q_1^{-1}(y_0) \cap \tilde{X}$ is a finite set. Hence, we have $\dim X_H = \dim \tilde{X}$. The equality $\dim \tilde{X} = \dim X$ is obvious by definition.

Step 2. $L_y \cap (X_H)_c \neq \phi$ for any point y in X_H.

Indeed, by Lemma 3. 3, we have $X_H \cdot \omega_{a_0, a_1, \cdots, a_{d-1}} = 0$. Hence, by the induction hypothesis on d and by the result of Step 1, we have the required assertion.

Step 3. $\dim Y_0 \leqslant \dim(\cup_{x \in X_0} L_x) \leqslant \dim X - d - 1 + d = \dim X - 1$.

Indeed, for any point y in $X_{H,0}$, there exists a point x in X such that $L_y \subset L_x$. Hence, $Y_0 = (\cup_{y \in X_{H,0}} L_y) \subset (\cup_{x \in X_0} L_x)$.

Step 4. $\dim Y = \dim X + d - 1$.

Indeed, we have $(\cup_{x \in X - X_0} L_x) \cap H \subset Y \subset (\cup_{x \in X} L_x) \cap H$ by definitions. Consider the following diagram.

$$\begin{tikzcd}[column sep=small] & Fl(n;d,0) \arrow[dl, "p_3"'] \arrow[dr, "q_3"] & \\ Gr(n,d) \supset X \supset X_0 & & Gr(n,0) = \boldsymbol{P}^n. \end{tikzcd}$$

By our assumption $L_{x_0} \cap X_c = \phi$ for a point x_0 in X, we see that $p_3^{-1}(X) \cap q_3^{-1}(P_0)$ is a finite set for a point P_0 in $\boldsymbol{P}^n$. Hence, we have $\dim(\cup_{x \in X} L_x) = \dim q_3 \circ p_3^{-1}(X) = \dim(\cup_{x \in X - X_0} L_x) = \dim q_3 \circ p_3^{-1}(X - X_0) = \dim X + d$. Therefore, since H is general, we have the required assertion.

Step 5. $L_y \cap Y_0 = \phi$ for some point y in X_H.

Indeed, if we assume the contrary, we are led to a contradiction as follows. Let y_1 be a generic point of X_H. We can take a point P_1 in $L_{y_1} \cap Y_0$ by our assumption in this step. We denote by Z the closure of (y_1, P_1) in $Gr(n-1, d-1) \times H$. Consider the diagram (3). Since $\pi_2 \circ \pi_1^{-1}(Z) = \{(Q, P) \in H \times H \mid$ *there exists a specialization* (y, P) *of* (y_1, P_1) *such that* $L_y \ni Q\}$, we have, in particuiar, $P \in L_y \cap Y_0$ for a point (Q, P) in $\pi_2 \circ \pi_1^{-1}(Z)$. Hence, we have $s \circ \pi_2 \circ \pi_1^{-1}(Z) = Y$. Therefore, we see that $r(\pi_2 \circ \pi_1^{-1}(Z) - s^{-1}(Y_0)) \neq \phi$. Hence, we may assume that there exists a point P_0 in $r(\pi_2 \circ \pi_1^{-1}(Z) - s^{-1}(Y_0))$ such that $P_0 \in L_{y_0} \subset L_{x_0}$, where y_0 is the point in X_H mentioned in Step 1. Then, by our assumption in Case 2, we see that $P_0 \notin X_c$, that is, the set $\{x \in X \mid L_x \ni P_0\}$ is finite. Hence, we have $\dim(r^{-1}(P_0) \cap (\pi_2 \circ \pi_1^{-1}(Z) - s^{-1}(Y_0))) = d-1$. Therefore, we have

$$\begin{aligned}\dim(\pi_2 \circ \pi_1^{-1}(Z) - s^{-1}(Y_0)) &\leqslant \dim q(Z) + d - 1 \\ &\leqslant \dim Y_0 + d - 1 \\ &\leqslant \dim X + d - 2.\end{aligned}$$

The last inequality follows from the result of Step 3. On the other hand, from the result of Step 4, we have

$$\dim(\pi_2 \circ \pi_1^{-1}(Z) - s^{-1}(Y_0)) = \dim Y = \dim X + d - 1.$$

This contradicts to the above estimation.

Step 6. By the result of Step 5, and by our assumption in Case 2, there exists a point x_2 in X and a point y_2 in X_H such that $L_{y_2} = L_{x_2} \cap H$, $L_{x_2} \cap X_c = \phi$, and $L_{y_2} \cap Y_0 = \phi$. By the induction hypothesis on d, there exists a point P_2 in $L_{y_2} \cap (X_H)_c$, that is, the set $\{y \in X_H \mid L_y \ni P_2\}$ is infinite. Since $P_2 \notin Y_0$, the set $\{y \in X_H \mid L_y \ni P_2\}$ is contained in $X_H - X_{H,0}$. Since $(X - X_0) \to (X_H - X_{H,0})$ is a surjective morphism and since the set $\{x \in X \mid L_x \ni P_2\}$ is the inverse image of the set $\{y \in X_H \mid L_y \ni P_2\}$, we see that the set $\{x \in X \mid L_x \ni P_2\}$ is infinite. This contradicts to $L_{x_2} \cap X_c = \phi$. Q.E.D.

Theorem 3. 1 follows immediately from Lemma 3. 5.

Pemark 3. 6. Probably the following statement (indicating more geometrical meaning) will be valid.

Let X be a subvariety of $Gr(n, d)$ of dimension $m \geqslant 2$. Assume that $X \cdot \omega_{a_0, a_1, \cdots, a_d} = 0$ for some Schubert cycle $\omega_{a_0, a_1, \cdots, a_d}$ of codimension $\sum_{i=0}^{d} a_i \leqslant m$. Then, for any point x_0 in X, there exists a curve C contained in X such that C goes through the point x_0 and that $\cap_{x \in C} L_x \neq \phi$.

But our dimension-theoretic argument is too rough to verify the above statement.

Corollary 3. 7. *Let X be a non-singular projective variety of dimension m defined over the complex number field* **C**. *Let E be a vector bundle of rank r on X. Suppose that E is ample and that, in addition, E is generated by its global sections. Then, E is numerically positive.*

Proof. Let q be an integer with $1 \leqslant q \leqslant m = \dim X$. Let Y be a q-dimensional subvariety of X. Let F be a quotient vector bundle of rank $d+1$ of $E|Y$ with $F \neq 0$. Let $P(T)$ be a homogeneous positive polynomial in $\Pi(d+1)$ of degree q with $P(T) \neq 0$. Since E is generated by its global sections, $E|Y$ and hence F is generated by its global sections. Hence, we have a morphism

$$f: Y \to Gr(n, d) \text{ with } F = f^*(\check{S}),$$

where $\check{S}$ is the dual of the universal subbandle S on $Gr(n, d)$. Since we may assume that $n - d \geqslant m$, we see that

$$\begin{array}{ccc} (I(d+1)\underset{Q}{\otimes} \boldsymbol{C})_q & \hookrightarrow & H^{2q}(Gr(n,d),\boldsymbol{C}) \\ \Cup & & \Cup \\ P(T) & \longmapsto & (\textit{the cohomology class of } P\left(\frac{\sqrt{-1}}{2\pi}\Theta_{\check{S}}\right)). \end{array}$$

Hence, we see that the cohomology class of $P\left(\frac{\sqrt{-1}}{2\pi}\Theta_{\check{S}}\right)$ is not zero. By Theorem 2. 1, the cohomology class of $P\left(\frac{\sqrt{-1}}{2\pi}\Theta_F\right)$ can be expressed as

$$\begin{aligned} &\textit{the cohomolgy class of } P\left(\frac{\sqrt{-1}}{2\pi}\Theta_F\right) \\ &= f^*\left(\textit{the cohomology class of } P\left(\frac{\sqrt{-1}}{2\pi}\Theta_{\check{S}}\right)\right) \\ &= \textit{the cohomology class of } \textstyle\sum_{a_0+a_1+\cdots+a_d=q} \alpha_{a_0,a_1,\cdots,a_d} f^*\omega_{a_0,a_1,\cdots,a_d}, \end{aligned}$$

where every coefficient $\alpha_{a_0,a_1,\cdots,a_d} \geqslant 0$ and they are not all zero. Since E is ample, $E|Y$ is ample and hence F is ample. Therefore, the morphism f is finite. Hence, $\check{S}|f(Y)$ is ample. Applying Theorem 3. 1 for $f(Y)$, we have that $f(Y)\cdot\omega_{a_0,a_1,\cdots,a_d}>0$ for any Schubert cycle $\omega_{a_0,a_1,\cdots,a_d}$ of codimension $\sum_{i=0}^{d} a_i = q$. Therefore, we see that

$$\begin{aligned} \int_Y P\left(\frac{\sqrt{-1}}{2\pi}\Theta_F\right) &= Y\cdot\left(\sum a_{a_0,a_1,\cdots,a_d} f^*\omega_{a_0,a_1,\cdots,a_d}\right) \\ &= \sum \alpha_{a_0,a_1,\cdots,a_d} Y\cdot f^*\omega_{a_0,a_1,\cdots,a_d} \\ &= \frac{1}{e}\sum \alpha_{a_0,a_1,\cdots,a_d} f_*(Y)\cdot\omega_{a_0,a_1,\cdots,a_d} \\ &> 0, \end{aligned}$$

where e is the mapping degree of the morphism f. Q.E.D.

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